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The maximum sum and product of sizes of cross-intersecting families

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Abstract

A family $\mathcal{A}$ of sets is said to be *t-intersecting* if any two distinct sets in $\mathcal{A}$ have at least t common elements. Families $\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_k$ are said to be *cross-t-intersecting* if for any i and j in $\{1, 2, ..., k\}$ with $i \neq j$, any set in $\mathcal{A}_i$ intersects any set in $\mathcal{A}_j$ on at least t elements. We present the following result: For any finite family $\mathcal{F}$ that has at least one set of size at least t, there exists an integer $k_0 \leq |\mathcal{F}|$ such that for any $k \geq k_0$, both the sum and product of sizes of k cross-t-intersecting sub-families $\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_k$ (not necessarily distinct or non-empty) of $\mathcal{F}$ are maxima if $\mathcal{A}_1 = \mathcal{A}_2 = ... = \mathcal{A}_k = \mathcal{L}$ for some largest t-intersecting sub-family $\mathcal{L}$ of $\mathcal{F}$. We also prove that if $t = 1$ and $\mathcal{F}$ is the family of all subsets of a set X, then the result holds with $k_0 = 2$ and $\mathcal{L}$ consisting of all subsets of X which contain a fixed element of X.

Keywords: cross-intersecting families, extremal set theory, intersecting families.

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1 Introduction

Unless otherwise stated, we use small letters such as x to denote elements of a set or positive integers, capital letters such as X to denote sets, and calligraphic letters such as $\mathcal{F}$ to denote *families* (that is, sets whose elements are sets themselves). Unless specified, sets and families are taken to be finite and may be the *empty set* $\emptyset$. An *r-set* is a set of size r, that is, a set having exactly r elements. For any integer $n \geq 1$, $[n]$ denotes the set $\{1, ..., n\}$ of the first n positive integers.

A family $\mathcal{A}$ is said to be *t-intersecting* if any two distinct sets in $\mathcal{A}$ intersect on at least t elements. Families $\mathcal{A}_1, ..., \mathcal{A}_k$ are said to be *cross-t-intersecting* if for any distinct i and j in $[k]$, any set in $\mathcal{A}_i$ intersects any set in $\mathcal{A}_j$ on at least t elements.

Let $\binom{[n]}{r}$ denote the family of all r-subsets of $[n]$. The classical Erdős-Ko-Rado (EKR) Theorem [6] says that if n is sufficiently larger than $r \geq t$, then a t-intersecting sub-family $\mathcal{A}$ of $\binom{[n]}{r}$ has size at most $\binom{n-t}{r-t}$, which is the number of sets in the trivial t-intersecting sub-family of $\binom{[n]}{r}$ consisting of those sets having $[t]$ as a subset. The EKR Theorem inspired a wealth of results of this kind, that is, results that establish how large a system of sets can be under certain intersection conditions; the survey papers [5,7] are recommended.

The main result presented here (Theorem 2.1 below) relates both the maximum sum and the maximum product of sizes of $k \geq 2$ cross-t-intersecting sub-families of a family $\mathcal{F}$ to the maximum size of an intersecting sub-family of $\mathcal{F}$ when k is not smaller than a certain value depending on $\mathcal{F}$ and t. It gives the maximum sum and the maximum product in terms of the size of a largest t-intersecting sub-family.

2 Cross-intersecting families

For any non-empty family $\mathcal{F}$, let $\alpha(\mathcal{F})$ denote the size of a largest set in $\mathcal{F}$. Suppose $\alpha(\mathcal{F}) < t$, and let $\mathcal{A}_1, ..., \mathcal{A}_k$ ($k \geq 2$) be sub-families of $\mathcal{F}$. Then $\mathcal{A}_1, ..., \mathcal{A}_k$ are cross-t-intersecting if and only if at most one of them is non-empty (since no set in $\mathcal{F}$ intersects itself or another set in $\mathcal{F}$ on at least t elements). Thus, if $\mathcal{A}_1, ..., \mathcal{A}_k$ are cross-t-intersecting, then the product of their sizes is 0 and the sum of their sizes is at most the size $|\mathcal{F}|$ of $\mathcal{F}$ (which is attained if and only if one of them is $\mathcal{F}$ and the others are all empty). This completely solves our problem for the case $\alpha(\mathcal{F}) < t$. We now address the case $\alpha(\mathcal{F}) \geq t$.

For any family $\mathcal{A}$, let $\mathcal{A}^{t,+}$ be the (t-intersecting) sub-family of $\mathcal{A}$ given by

$\mathcal{A}^{t,+} = \{A \in \mathcal{A}\colon |A \cap B| \geq t \text{ for any } B \in \mathcal{A}\backslash\{A\}\}$, and let $\mathcal{A}^{t,-} = \mathcal{A}\backslash\mathcal{A}^{t,+}$. In simple terms, a set A in $\mathcal{A}$ is in $\mathcal{A}^{t,-}$ if and only if there exists another set B in $\mathcal{A}$ such that A and B do not have t common elements, otherwise A is in $\mathcal{A}^{t,+}$. The definitions of $\mathcal{A}^{t,+}$ and $\mathcal{A}^{t,-}$ are generalisations of the definitions of $\mathcal{A}^*$ and $\mathcal{A}'$ in [1,2,3,4]; $\mathcal{A}^* = \mathcal{A}^{1,+}$ and $\mathcal{A}' = \mathcal{A}^{1,-}$.

Let $l(\mathcal{F}, t)$ denote the size of a largest t-intersecting sub-family of a *non-empty* family $\mathcal{F}$. For any sub-family $\mathcal{A}$ of $\mathcal{F}$, we define

$$\beta(\mathcal{F}, t, \mathcal{A}) = \begin{cases} \frac{l(\mathcal{F},t)-|\mathcal{A}^{t,+}|}{|\mathcal{A}^{t,-}|} & \text{if } \mathcal{A}^{t,-} \neq \emptyset \\ \frac{l(\mathcal{F},t)}{|\mathcal{F}|} & \text{if } \mathcal{A}^{t,-} = \emptyset; \end{cases}$$

so $|\mathcal{A}^{t,+}| + \beta(\mathcal{F}, t, \mathcal{A})|\mathcal{A}^{t,-}| \leq l(\mathcal{F}, t)$ (even if $\mathcal{A}^{t,-} = \emptyset$ because $|\mathcal{A}^{t,+}| \leq l(\mathcal{F}, t)$ since $\mathcal{A}^{t,+}$ is t-intersecting). We now define

$$\beta(\mathcal{F}, t) = \min\{\beta(\mathcal{F}, t, \mathcal{A})\colon \mathcal{A} \subseteq \mathcal{F}\}.$$

Therefore,

$$|\mathcal{A}^{t,+}| + \beta(\mathcal{F}, t)|\mathcal{A}^{t,-}| \leq l(\mathcal{F}, t) \quad \text{for any } \mathcal{A} \subseteq \mathcal{F}. \tag{1}$$

One can also show that

$$\beta(\mathcal{F}, t) = \max\left\{c \in \mathbb{R}\colon c \leq \frac{l(\mathcal{F}, t)}{|\mathcal{F}|},\ |\mathcal{A}^{t,+}| + c|\mathcal{A}^{t,-}| \leq l(\mathcal{F}, t)\ \forall\, \mathcal{A} \subseteq \mathcal{F}\right\} \tag{2}$$

and that

$$\frac{1}{|\mathcal{F}|} \leq \beta(\mathcal{F}, t) \leq \frac{l(\mathcal{F}, t)}{|\mathcal{F}|} \tag{3}$$

Thus, for any non-empty family $\mathcal{F}$, we can define $\kappa(\mathcal{F}, t) = \frac{1}{\beta(\mathcal{F},t)}$ and we have $\kappa(\mathcal{F}, t) \leq |\mathcal{F}|$. We can now state our main result.

Theorem 2.1 *Let $\mathcal{A}_1, ..., \mathcal{A}_k$ be cross-t-intersecting sub-families of a family $\mathcal{F}$ with $\alpha(\mathcal{F}) \geq t$. If $k \geq \kappa(\mathcal{F}, t)$ then*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq k.l(\mathcal{F}, t) \quad \text{and} \quad \prod_{i=1}^{k} |\mathcal{A}_i| \leq (l(\mathcal{F}, t))^k,$$

and if $k > \kappa(\mathcal{F}, t)$ then both bounds are attained if and only if $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$ for some largest t-intersecting sub-family $\mathcal{L}$ of $\mathcal{F}$.

If $k < \kappa(\mathcal{F}, t)$ then the sum inequality above does not hold for $\mathcal{A}_1, ..., \mathcal{A}_k$ with a maximum value of $\sum_{i=1}^{k} |\mathcal{A}_i|$.

Theorem 2.2 *Let $\mathcal{F}$ be a family with $\alpha(\mathcal{F}) \geq t$. Let $\mathcal{A}_1, ..., \mathcal{A}_k$ be cross-t-intersecting sub-families of $\mathcal{F}$ such that $\sum_{i=1}^{k} |\mathcal{A}_i|$ is a maximum. Then:*
(i) $\sum_{i=1}^{k} |\mathcal{A}_i| = k.l(\mathcal{F}, t)$ if $k \geq \kappa(\mathcal{F}, t)$;
(ii) $\sum_{i=1}^{k} |\mathcal{A}_i| > k.l(\mathcal{F}, t)$ if $k < \kappa(\mathcal{F}, t)$.

An immediate consequence of the two results above is that the maximum product is $(l(\mathcal{F}, t))^k$ if the maximum sum is $k.l(\mathcal{F}, t)$.

Corollary 2.3 *Let $\mathcal{F}$ be a family with $\alpha(\mathcal{F}) \geq t$. If $\sum_{i=1}^{k} |\mathcal{A}_i| \leq k.l(\mathcal{F}, t)$ for any cross-t-intersecting sub-families $\mathcal{A}_1, ..., \mathcal{A}_k$ of $\mathcal{F}$, then $\prod_{i=1}^{k} |\mathcal{A}_i| \leq (l(\mathcal{F}, t))^k$ for any cross-t-intersecting sub-families $\mathcal{A}_1, ..., \mathcal{A}_k$ of $\mathcal{F}$.*

In other words, if the configuration $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$ (where $\mathcal{L}$ is as in Theorem 2.1) gives a maximum sum, then $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$ also gives a maximum product. The converse is not true, that is, $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$ may still give a maximum product when $k < \kappa(\mathcal{F}, t)$. For example, the main result in [9] tells us that the product of sizes of 2 cross-1-intersecting sub-families $\mathcal{A}_1$ and $\mathcal{A}_2$ of $\binom{[n]}{r}$ is a maximum if $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{L}$ for some largest 1-intersecting sub-family $\mathcal{L}$ of $\binom{[n]}{r}$, where $\mathcal{L} = \binom{[n]}{r}$ if $r > n/2$, and by the EKR Theorem [6], $\mathcal{L}$ is of size $\binom{n-1}{r-1}$ (the size of the trivial 1-intersecting sub-family $\{A \in \binom{[n]}{r} : 1 \in A\}$ of $\binom{[n]}{r}$) if $r \leq n/2$. It follows easily that the product of sizes of any $k \geq 2$ cross-1-intersecting sub-families $\mathcal{A}_1, ..., \mathcal{A}_k$ of $\binom{[n]}{r}$ is a maximum if $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$. Now in [1] it is shown that for $n \geq 2r$, we have $\beta\left(\binom{[n]}{r}, 1\right) = \frac{|\mathcal{L}|}{\binom{n}{r}} = \frac{r}{n}$ and hence $\kappa\left(\binom{[n]}{r}, 1\right) = \frac{n}{r}$. Thus, for any n, r and k with $2 \leq k < \frac{n}{r}$, we have $k < \kappa\left(\binom{[n]}{r}, 1\right)$ and yet the configuration $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$ gives a maximum product.

However, just like the threshold $\kappa(\mathcal{F}, t)$ for the maximum sum part of Theorem 2.1 cannot be improved (by Theorem 2.2), the threshold $\kappa(\mathcal{F}, t)$ for the maximum product part of Theorem 2.1 can neither be improved in general, as we will now give a (geometrical) construction of a family $\mathcal{F}$ such that for any $k < \kappa(\mathcal{F}, t)$, the product of k cross-t-intersecting sub-families $\mathcal{A}_1, ..., \mathcal{A}_k$ of $\mathcal{F}$ is not a maximum if $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$.

Example 2.4 *Let $p \geq 3$ be an integer. Let $m_1, ..., m_p$ and $c_{1,1}, ..., c_{1,p}$, $c_{2,1}, ...,$ $c_{2,p}$, ..., $c_{p,1}, ..., c_{p,p}$ be distinct real numbers. For each $i, j \in [p]$, let $L_{i,j}$ be the straight line in $\mathbb{R}^2$ arising from the function $y \colon \mathbb{R} \to \mathbb{R}$ defined by $y(x) = m_i x + c_{i,j}$. For each $i, j \in [p]$, let $A_{i,j}$ be the set of all points (i.e. co-ordinates)* *of intersection of* $L_{i,j}$ *with all the other lines* $L_{i',j'}$*, i.e.*

$$A_{i,j} = \{(a,b) \in \mathbb{R}^2 \colon \exists\, i', j' \in [p],\ (i'j') \neq (i,j),\ \text{such that}\ L_{i,j}\ \text{intersects}\ L_{i',j'}\ \text{on}\ (a,b)\}.$$

Let $(a_1, b_1), ..., (a_s, b_s)$ *be the distinct co-ordinates in the set* $\bigcup_{i=1}^{p} \bigcup_{j=1}^{p} A_{i,j}$ *of all points of intersection of these lines, and let* $T_{(a_1,b_1)}, ..., T_{(a_s,b_s)}$ *be disjoint sets of size* t*. For each* $i, j \in [p]$*, let* $B_{i,j} = \bigcup_{(a,b)\in A_{i,j}} T_{(a,b)}$*; so* $B_{i,j}$ *is simply the set obtained by replacing each point* (a, b) *in* $A_{i,j}$ *by the corresponding* t*-set* $T_{(a,b)}$*. For each* $i \in [p]$*, let* $\mathcal{B}_i = \{B_{i,1}, ..., B_{i,p}\}$*. Now let* $\mathcal{B} = \bigcup_{i=1}^{p} \mathcal{B}_i = \{B_{i,j} \colon i, j \in [p]\}$*.*

Theorem 2.5 *Let* $\mathcal{B}$ *be as in Example 2.4. Let* $\mathcal{L}$ *be a largest* t*-intersecting sub-family of* $\mathcal{B}$*. Let* $\mathcal{A}_1, ..., \mathcal{A}_k$ *be cross-*t*-intersecting sub-families of* $\mathcal{B}$*. Then:*
(i) $\kappa(\mathcal{B}, t) = |\mathcal{L}| = p$*,*
(ii) if $k \geq \kappa(\mathcal{B}, t)$ *and* $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$*, then* $\prod_{i=1}^{k} |\mathcal{A}_i|$ *is a maximum.*
(iv) if $k < \kappa(\mathcal{B}, t)$ *and* $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$*, then* $\prod_{i=1}^{k} |\mathcal{A}_i|$ *is not a maximum.*

3 The special case when $t = 1$ and $\mathcal{F}$ is a power set

Let $2^{[n]}$ denote the power set of $[n]$, i.e. the family of all subsets of $[n]$. We determine $\beta(\mathcal{F}, 1)$ and, by applying Theorem 2.1, we then solve the cross-1-intersection problem for $2^{[n]}$. One of the basic results in extremal set theory is that $l(2^{[n]}, 1) = 2^{n-1}$ (see [6]), and this is generalised by our next result.

Lemma 3.1 *If* $\mathcal{F} = 2^{[n]}$ *then* $\beta(\mathcal{F}, 1) = \frac{l(\mathcal{F},1)}{|\mathcal{F}|} = \frac{1}{2}$*.*

Proof Let $\mathcal{A} \subseteq \mathcal{F} = 2^{[n]}$. Let $\mathcal{B} = \{[n]\backslash A \colon A \in \mathcal{A}^{1,+}\}$. So $|\mathcal{B}| = |\mathcal{A}^{1,+}|$. For any $B \in \mathcal{B}$, $B = [n]\backslash A$ for some $A \in \mathcal{A}^{1,+}$, and hence, by definition of $\mathcal{A}^{1,+}$, $B \notin \mathcal{A}$ since $|A \cap B| = 0$. So $\mathcal{A}$ and $\mathcal{B}$ are disjoint sub-families of $\mathcal{F}$. Thus,

$$2|\mathcal{A}^{1,+}| + |\mathcal{A}^{1,-}| = |\mathcal{A}^{1,+}| + |\mathcal{B}| + |\mathcal{A}^{1,-}| = |\mathcal{A}| + |\mathcal{B}| = |\mathcal{A} \cup \mathcal{B}| \leq |\mathcal{F}| = 2^n$$

and hence, dividing throughout by 2, we get $|\mathcal{A}^{1,+}| + \frac{1}{2}|\mathcal{A}^{1,-}| \leq 2^{n-1}$. It follows that a 1-intersecting sub-family of $\mathcal{F}$ has size at most 2^{n-1} (as $\mathcal{A} = \mathcal{A}^{1,+}$ if $\mathcal{A}$ is 1-intersecting), and this bound is attained by the trivial 1-intersecting sub-family $\{A \in \mathcal{F} \colon 1 \in A\}$; so $l(\mathcal{F}, 1) = 2^{n-1}$ and $\frac{l(\mathcal{F},1)}{|\mathcal{F}|} = \frac{1}{2}$. So we have $|\mathcal{A}^{1,+}| + \frac{l(\mathcal{F},1)}{|\mathcal{F}|}|\mathcal{A}^{1,-}| \leq l(\mathcal{F}, 1)$. By (2), $\beta(\mathcal{F}, 1) = \frac{l(\mathcal{F},1)}{|\mathcal{F}|} = \frac{1}{2}$. □

Theorem 3.2 *Let $\mathcal{A}_1, ..., \mathcal{A}_k$ be cross-1-intersecting sub-families of $2^{[n]}$. Then*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq k2^{n-1} \quad \text{and} \quad \prod_{i=1}^{k} |\mathcal{A}_i| \leq 2^{k(n-1)},$$

and both bounds are attained if $\mathcal{A}_1 = ... = \mathcal{A}_k = \{A \in 2^{[n]} : 1 \in A\}$. If $k > 2$ then both bounds are attained if and only if $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$ for some largest 1-intersecting sub-family $\mathcal{L}$ of $\mathcal{F}$.

Proof By Lemma 3.1, $\kappa\left(2^{[n]}, 1\right) = 2$. The result follows by Theorem 2.1. □

We point out that $\beta(\mathcal{F}, 1)$ is determined precisely, and shown to be also equal to $\frac{l(\mathcal{F},1)}{|\mathcal{F}|}$, for the case when $\mathcal{F}$ is $\binom{[n]}{r}$ ([1]), the family of permutations of $[n]$ ([2]), the family of r-partial permutations of $[n]$ ([3]), and the family of signed r-sets on $[n]$ ([4]). Moreover, for each of these families, the problem of maximising the sum of sizes is solved completely for $t = 1$.

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